4 Simple harmonic motion and waves

4.1 Kinematics of simple harmonic motion

Assessment statements

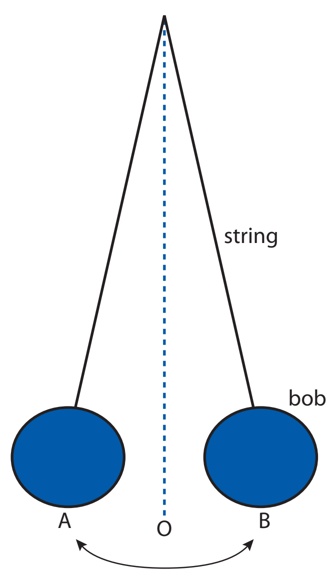
4.1.1 Describe examples of oscillations.
4.1.2 Define the terms *displacement*, *amplitude*, *frequency*, *period* and *phase difference*.
4.1.3 Define *simple harmonic motion* (SHM) and state the defining equation as $a = -\omega^2 x$.
4.1.4 Solve problems using the defining equation for SHM.
4.1.5 Apply the equations $v = v_0 \sin \omega t$, $v = v_0 \cos \omega t$, $v = \pm\omega\sqrt{(x_0^2 - x^2)}$, $x = x_0 \cos \omega t$ and $x = x_0 \sin \omega t$ as solutions to the defining equation for SHM.
4.1.6 Solve problems, both graphically and by calculation, for acceleration, velocity and displacement during SHM.

A swing is an example of oscillatory motion.

Oscillations

In this section we will derive a mathematical model for an oscillating or vibrating body. There are many different examples of naturally occurring oscillations but they don't all have the same type of motion. We are going to consider the simplest form of oscillation: simple harmonic motion. The most common example of this is a pendulum (Figure 4.1). Before we start to model this motion, we need to define some new terms and quantities.

Figure 4.1 The simple pendulum swings from A to B and back.



Cycle

One cycle is defined as one complete oscillation of the pendulum (A–B–A). The term *cycle* is also used to describe circular motion; one cycle is one complete circle or 2π radians.

Equilibrium position

The equilibrium position is the position where the pendulum bob would rest if not disturbed – this is position O.

Amplitude (x_0)

The amplitude is defined as the maximum displacement from the equilibrium position, this is distance OB or OA.

Unit: metre

Measuring time period

To reduce the uncertainties when measuring the time period of an oscillation, it is best to measure the time for many oscillations e.g. 10. The time period is found by dividing this time by the number of oscillations.

Time period (T)

The time period is the time taken for one complete cycle.

Unit: second

Frequency (f)

The frequency is the number of cycles that the pendulum makes per unit time. This is equal to 1/time period.

Unit: s^{-1} or hertz (Hz)

Angular frequency (ω)

The angular frequency is found by multiplying f by 2π ($\omega = 2\pi f$). This quantity is normally used when describing circular motion. An angular frequency of 2π rads s^{-1} means that a body makes one revolution per second. However, it is also used to describe an oscillation, 2π being equivalent to one complete cycle.

Unit: s^{-1} or hertz (Hz)

Worked example

A pendulum completes 10 swings in 8 s. What is the angular frequency?

Solution

There are 10 swings in 8 seconds, so each swing takes 0.8 s.

Time period = 0.8 s.

Frequency $= \frac{1}{T} = \frac{1}{0.8} = 1.25$ Hz

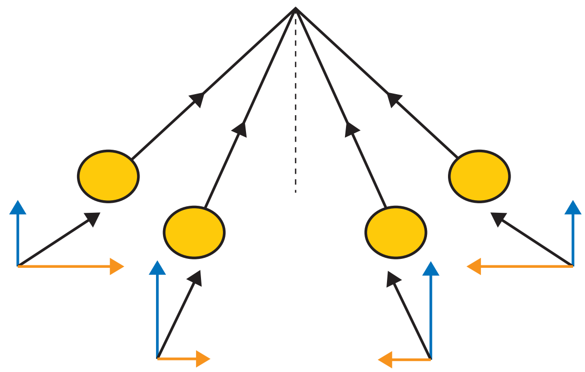
Angular frequency $\omega = 2\pi f = 2\pi \times 1.25 = \mathbf{7.8\ rad\ s^{-1}}$

Analysing oscillations

To make a model of oscillatory motion, we will analyse two different oscillations and see if there are any similarities.

The pendulum

Figure 4.2 As the angle increases, the horizontal component of tension increases, but is always pointing towards the centre.

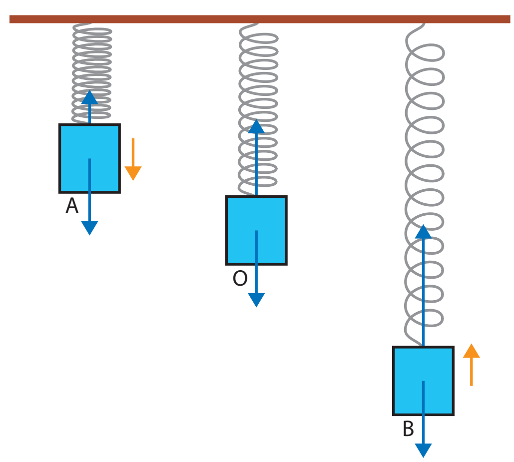


When a pendulum bob is pushed to one side and released, it will swing back down. The reason for this can be understood by drawing the forces acting on the bob. In Figure 4.2, we can see that when the string makes an angle to the vertical, the tension has a component in the horizontal direction; this component causes the bob to accelerate back towards the middle. As the bob swings down, the angle of the string gets smaller, and the horizontal component decreases. The horizontal acceleration of the bob is proportional to the horizontal force, so we can therefore deduce that the horizontal acceleration is proportional to the displacement from the lowest point.

When the ball reaches the lowest position, it is travelling at its maximum speed. It passes through this position and continues to swing up on the other side. The horizontal component of the tension is now acting in the other direction. This is in the *opposite* direction to the motion so will slow the bob down. We can conclude that no matter where the bob is, its acceleration is always directed towards O.

Mass on a spring

Figure 4.3 The tension increases as the spring is stretched. The resultant (red) increases with increased distance from the centre and is always directed towards the centre.



If a mass hanging on the end of a spring is lifted up and released, it will bounce up and down as in Figure 4.3. The forces acting on the mass are *weight* and the *tension in the spring*. The weight is always the same but the tension depends on how far the spring is stretched. (When you stretch a spring, the tension is proportional to the extension).

At A, the spring is short, so the tension will be small; the weight will therefore be bigger than the tension, so the resultant force will be downwards.

As the mass passes through the middle point, the forces will be balanced.

At B, the spring is stretched, so the tension is large; the tension will therefore be greater than the weight, so the resultant force will be upwards.

Again we can see that the acceleration is proportional to the displacement from the central point and always directed towards it.

This type of motion is called *simple harmonic motion* or SHM.

SHM

The acceleration is proportional to the distance from a fixed point.
The acceleration is always directed towards a fixed point.

Exercises

1 State whether the following are examples of simple harmonic motion.

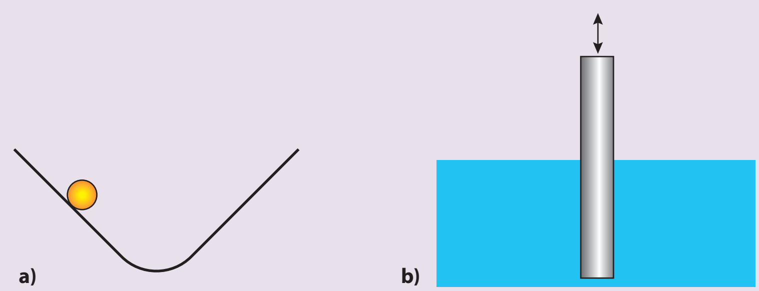


Figure 4.4

(a) A ball rolling up and down on a track (Figure 4.4a).
(b) A cylindrical tube floating in water when pushed down and released (Figure 4.4b).
(c) A tennis ball bouncing back and forth across the net.
(d) A bouncing ball.

2 A pendulum completes 20 swings in 12 s. What is
(a) the frequency?
(b) the angular frequency?

Graphical treatment

To analyse the oscillation further, we can plot graphs for the motion. In this example, we will consider a mass on a spring, but we could choose any simple harmonic motion.

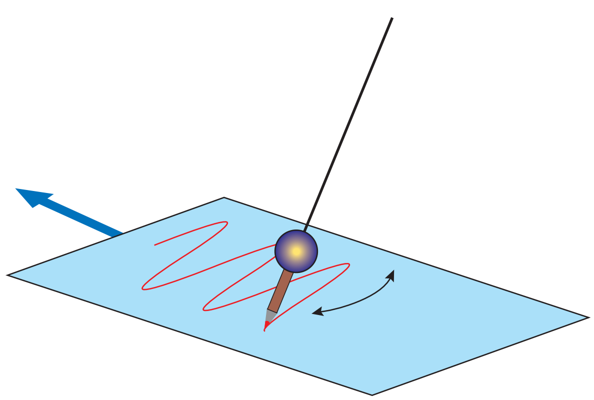


Figure 4.5 You can plot a displacement–time graph by attaching a pen to a pendulum and moving paper beneath it at a constant velocity.

To see how the equation fits the graph we can put some numbers into the equation.
In this example, the time period = 4 s
Therefore $f = \frac{1}{4} = 0.25$ Hz
Angular frequency $= 2\pi f = 0.5\pi$
So displacement $= 2\cos(0.5\pi t)$
Calculating displacement at different times gives:
$t = 1$ s $y = 2\cos(\pi/2) = 0$ cm
$t = 2$ s $y = 2\cos(\pi) = -2$ cm
$t = 3$ s $y = 2\cos(3\pi/2) = 0$ cm
$t = 4$ s $y = 2\cos(2\pi) = 2$ cm

Displacement–time

As before, O is the equilibrium position and we will take this to be our position of zero displacement. Above this is positive displacement and below is negative.

At A, the mass has maximum positive displacement from O.

At O, the mass has zero displacement from O.

At B, the mass has maximum negative displacement from O.

We can see that the shape of this displacement–time graph is a cosine curve.

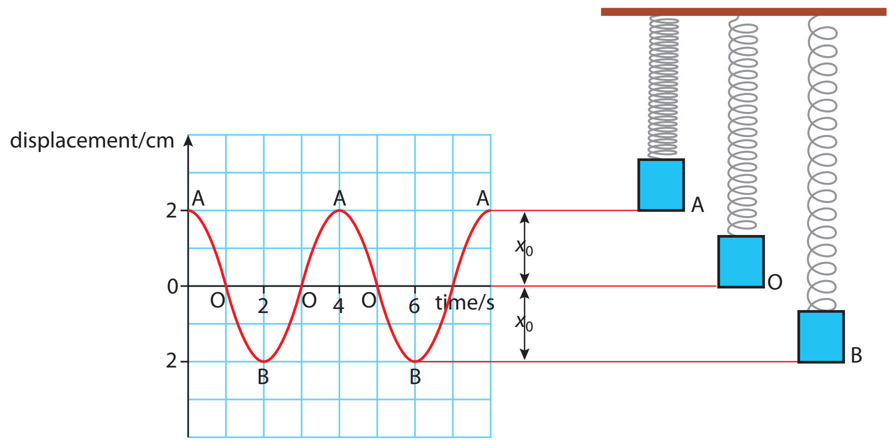


Figure 4.6 Displacement–time graph.

The equation of this line is $x = x_0 \cos \omega t$,

where x_0 *is* the maximum displacement and ω is the angular frequency.

Velocity–time

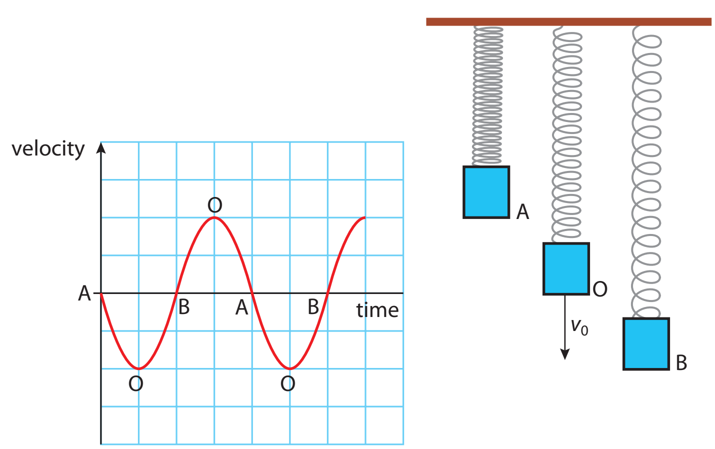
From the gradient of the displacement–time graph (Figure 4.6), we can calculate the velocity.

At A, gradient = 0 so velocity is zero.

At O, gradient is negative and maximum, so velocity is down and maximum.

At B, gradient = 0 so velocity is zero.

Figure 4.7 Velocity–time graph.



The equation of this line is $v = -v_0 \sin \omega t$ where v_0 is the maximum velocity.

Acceleration–time

From the gradient of the velocity–time graph (Figure 4.7) we can calculate the acceleration.

At A, the gradient is maximum and negative so acceleration is maximum and downwards.

At O, the gradient is zero so acceleration is zero.

At B, the gradient is maximum and positive so the acceleration is maximum and upwards.

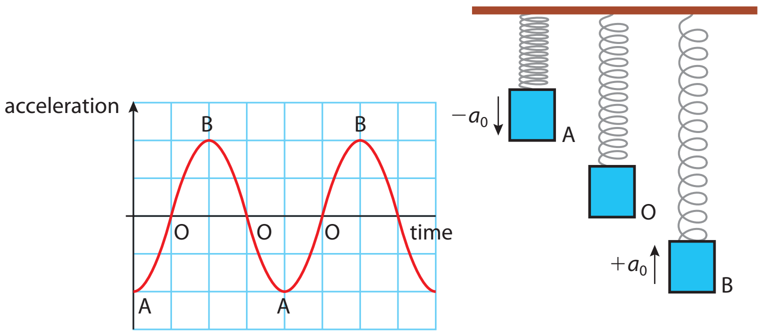


Figure 4.8 Acceleration–time graph.

The equation of this line is $a = -a_0 \cos \omega t$ where a_0 is this maximum acceleration. So $x = x_0 \cos \omega t$ and $a = -a_0 \cos \omega t$

When displacement increases, acceleration increases proportionally but in a negative sense; in other words: $a \propto -x$

We have confirmed that the acceleration of the body is directly proportional to the displacement of the body and always directed towards a fixed point.

Worked example

A mass on a spring is oscillating with a frequency 0.2 Hz and amplitude 3.0 cm. What is the displacement of the mass 10.66 s after it is released from the top?

Solution

$x = x_0 \cos \omega t.$ Since this is SHM

where x = displacement

x_0 = amplitude = 3 cm

ω = angular velocity = $2\pi f = 2\pi \times 0.2$

$= 0.4\pi$ Hz

t = time = 10.66 s

$x = 0.03 \times \cos(0.4\pi \times 10.66)$ Substituting into the formula

$x = 0.02$ m

= 2 cm

Radians
When calculating cos ωt, you must have your calculator set on radians.

Exercises

3 For the same mass on a spring in the example above, calculate the displacement after 1.55 s.

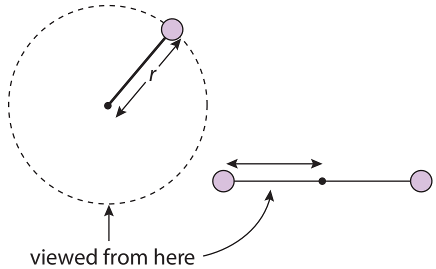
4 Draw a displacement time sketch graph for this motion.

5 A long pendulum has time period 10 s. If the bob is displaced 2 m from the equilibrium position and released, how long will it take to move 1 m?

6 As a mass on a spring travels upwards through the equilibrium position, its velocity is 0.5 m s^{-1}. If the frequency of the pendulum is 1 Hz what will the velocity of the bob be after 0.5 s?

SHM and circular motion

Figure 4.9 A short time after the ball starts moving, the radius makes an angle θ with the horizontal.



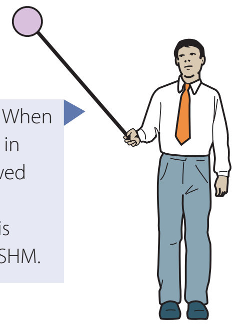


Figure 4.10 When a ball moving in a circle is viewed from the side, it looks like it is moving with SHM.

If we analyse the motion of the ball in Figure 4.9, we find that it is also SHM. The ball is travelling in a circle of radius x_0 with a constant speed v. The ball takes a time T to complete one revolution.

Let us consider the horizontal component of motion. We can write equations for this component.

Displacement

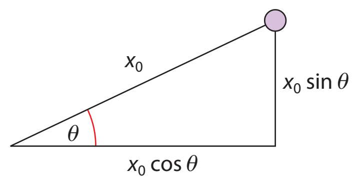


Figure 4.11 From the triangle we can see that *the horizontal displacement* $x = x_0 \cos\theta$.

Velocity

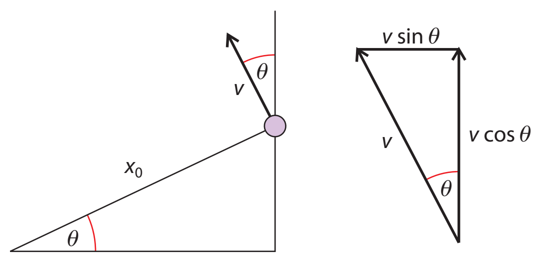


Figure 4.12 Horizontal velocity vectors.

The horizontal velocity $= -v \sin\theta$

But for circular motion $v = \omega r$ so in this case $v = \omega x_0$

So horizontal velocity $= -\omega x_0 \sin\theta$

Speed, $v = \frac{\text{distance}}{\text{time}}$

$= \frac{\text{circumference}}{\text{time period}}$

$= \frac{2\pi r}{T}$

But $\frac{2\pi}{T} = \omega$

So speed $= \omega r$

Centripetal acceleration $= \frac{v^2}{r}$

$= \frac{\omega^2 r^2}{r}$

$= \omega^2 r$

Acceleration

When bodies travel in a circle, they have an acceleration towards the centre (the centripetal acceleration) $a = \omega^2 r$. In this case, acceleration is $\omega^2 x_0$ since the radius is x_0.

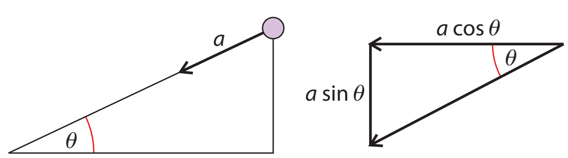


Figure 4.13 Horizontal acceleration vectors.

Horizontal component of acceleration $= -a \cos\theta$

But $a = \omega^2 x_0$

So horizontal acceleration $= -\omega^2 x_0 \cos\theta$

Now we have found that the displacement $x = x_0 \cos\theta$

So acceleration$= -\omega^2 x$

So the horizontal acceleration is proportional to the displacement, and is always directed towards the centre. In other words, the horizontal component of the motion is SHM. We have also found out that the constant of proportionality is ω^2.

Now we have concluded that this motion is SHM we can use the equations that we have derived to model all simple harmonic motions.

If you have done differentiation in maths then you will understand that if

displacement $x = x_0 \cos \omega t$

then velocity $\frac{dx}{dt} = -x_0\omega \sin \omega t$

and acceleration, $\frac{d^2x}{dt^2} = -x_0\omega^2 \cos \omega t$

This implies that $a = -\omega^2 x$

This is a much shorter way of deriving this result!

Equations for SHM

Displacement $= x_0 \cos \omega t$ (1)

Velocity $= -\omega x_0 \sin \omega t$ (2)

Acceleration $= -\omega^2 x_0 \cos \omega t$ (3)

We also know that $a = -\omega^2 x$

From Pythagoras, $1 = \sin^2 \theta + \cos^2 \theta$

So, $\sin \theta = \sqrt{1 - \cos^2 \theta}$ — Rearranging

Therefore $\sin \omega t = \sqrt{1 - \cos^2 \omega t}$ — Substituting for $\theta = \omega t$

Multiplying by ω gives $\omega x_0 \sin \omega t = \omega x_0 \sqrt{1 - \cos^2 \omega t}$

$\omega x_0 \sin \omega t = \omega \sqrt{x_0^2 - x_0^2 \cos^2 \omega t}$ — Taking x_0 into the square root

But from equation (1) $x_0^2 \cos^2 \omega t = x^2$

So $v = \omega \sqrt{x_0^2 - x^2}$ — Substituting

The maximum velocity is when the displacement is 0 so $x = 0$

Maximum velocity $= \omega x$

Worked example

1 A pendulum is swinging with a frequency of 0.5 Hz. What is the size and direction of the acceleration when the pendulum has a displacement of 2 cm to the right?

2 A pendulum bob is swinging with SHM at a frequency of 1Hz and amplitude 3 cm. At what position will the bob be moving with maximum velocity and what is the size of the velocity?

Solution

1 Assuming the pendulum is swinging with SHM, then we can use the equation $a = \omega^2 x$ to calculate the acceleration.

$\omega = 2\pi f = 2\pi \times 0.5 = \pi$

$a = -\pi^2 \times 0.02 = \mathbf{-0.197\ m\,s^{-2}}$ — Since $-$ve direction is to the left

2 $v = \omega \sqrt{x_0^2 - x^2}$ — Since the motion is SHM

This is maximum when $x = 0$ — This is when the pendulum swings through the central position

The maximum value $= \omega x_0$ where $\omega = 2\pi f = 2 \times \pi \times 1 = 2\pi\ \mathrm{rad\,s^{-1}}$

Maximum $v = 2\pi \times 0.03 = \mathbf{0.188\ m\,s^{-1}}$

Exercises

7 A long pendulum swings with a time period of 5 s and an amplitude of 2 m.

(a) What is the maximum velocity of the pendulum?

(b) What is the maximum acceleration of the pendulum?

8 A mass on a spring oscillates with amplitude 5 cm and frequency 2 Hz. The mass is released from its highest point. Calculate the velocity of the mass after it has travelled 1 cm.

9 A body oscillates with SHM of time period 2 s. What is the amplitude of the oscillation if its velocity is $1\ \mathrm{m\,s^{-1}}$ as it passes through the equilibrium position?

Summary

If a body oscillating with SHM has an angular frequency $\omega = 2\pi f$ and amplitude x_0 then its displacement (x), velocity (v) and acceleration (a) at any given time can be found from the following equations:

$x = x_0 \cos \omega t$

$v = -x_0 \omega \sin \omega t$

$a = -\omega^2 x_0 \cos \omega t$

In addition, at a given displacement x, the velocity and acceleration can be found from the following equations:

$v = \omega \sqrt{x_0^2 - x^2}$

Maximum velocity $= \omega x_0$

$a = -\omega^2 x$

The real pendulum

The pendulum is a classic example of SHM. However it is only SHM if the swings are very small (less than 10°). This is worth remembering if you ever do an experiment with a real pendulum.

4.2 Energy changes during simple harmonic motion (SHM)

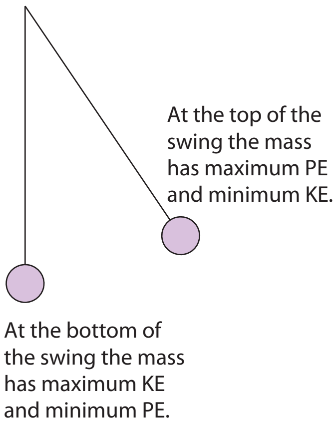
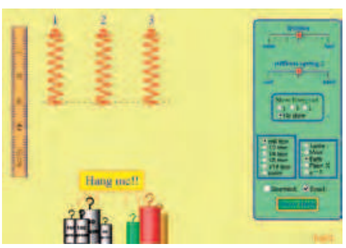


Figure 4.14 In the simple pendulum, energy is changing from one form to another as it moves.



To view the PhET *Masses and springs* simulation, visit heinemann.co.uk/hotlinks, enter the express code 4426P and click on Weblink 4.1.

Assessment statements

4.2.1 Describe the interchange between kinetic energy and potential energy during SHM.

4.2.2 Apply the expression $E_K = \frac{1}{2}m\omega^2(x_0^2 - x^2)$ for the kinetic energy of a particle undergoing SHM, $E_T = \frac{1}{2}m\omega^2 x_0^2$ for the total energy and $E_P = \frac{1}{2}m\omega^2 x^2$ for the potential energy.

4.2.3 Solve problems, both graphically and by calculation, involving energy changes during SHM.

If we once again consider the simple pendulum, we can see that its energy changes as it swings.

Kinetic energy

We have already shown that the velocity of the mass is given by the equation

$$v = \omega\sqrt{x_0^2 - x^2}$$

From definition, KE $= \frac{1}{2}mv^2$

Substituting: KE $= \frac{1}{2}m\omega^2(x_0^2 - x^2)$

KE is a maximum at the bottom of the swing where $x = 0$.

So $KE_{max} = \frac{1}{2}m\omega^2 x_0^2$

At this point the PE is zero.

Total energy

The total energy at any moment in time is given by:

total energy = KE + PE

So at the bottom of the swing:

total energy $= \frac{1}{2}m\omega^2 x_0^2 + 0 = \frac{1}{2}m\omega^2 x_0^2$

Since no work is done on the system, according to the law of conservation of energy, the total energy must be constant.

So total energy $= \frac{1}{2}m\omega^2 x_0^2$

Potential energy

Potential energy at any moment = total energy − KE

So PE $= \frac{1}{2}m\omega^2 x_0^2 - \frac{1}{2}m\omega^2(x_0^2 - x^2)$

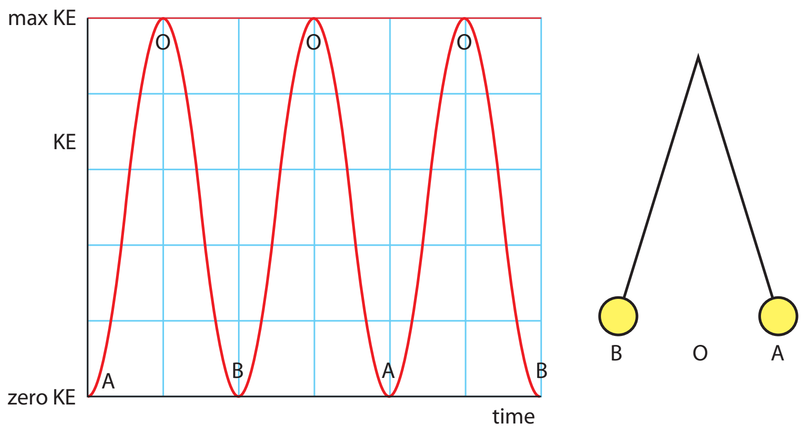
PE $= \frac{1}{2}m\omega^2 x^2$

Solving problems graphically

Kinetic energy

From previous examples we know that the velocity, $v = -v_0 \sin \omega t$

So $\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 \sin^2 \omega t$



The kinetic energy is a maximum when the bob is travelling fastest; this is at the bottom of the swing. At the top of the swing, the bob is stationary, so the KE is zero.

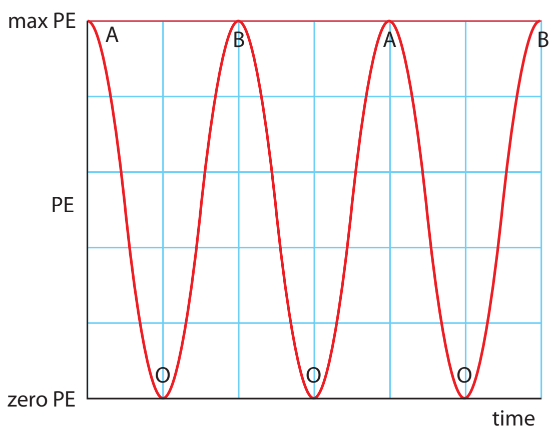
Figure 4.15 The graph of KE vs time is a $\sin^2$ curve.

Potential energy

The graph of PE can be found from PE $= \frac{1}{2} m\omega^2 x^2$

Since $x = x_0 \cos \omega t$

PE $\quad = \frac{1}{2}m\omega^2 x_0^2 \cos^2 \omega t = \frac{1}{2}mv_0^2 \cos^2 \omega t$

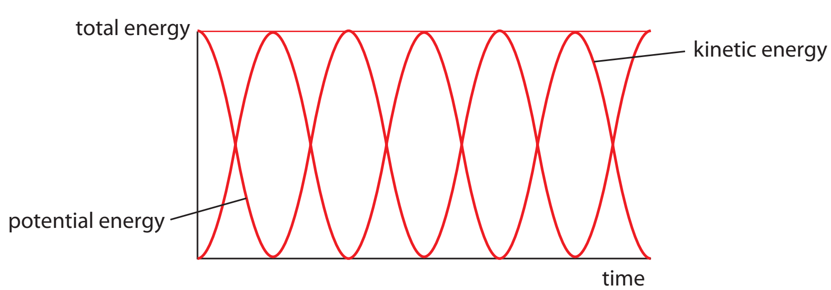


Potential energy
The potential energy is a minimum when the bob is at its lowest point; we take this to be zero. At the top of the swing, the potential energy is a maximum value.

Figure 4.16 The graph of PE vs time is a $\cos^2$ curve.

Total energy

If these two graphs are added together it gives a constant value, the total energy. (This might remind you of Pythagoras: $1 = \cos^2\theta + \sin^2\theta$.)



Total energy
If no energy is lost then the total energy is a constant value. When the bob is swinging, the energy continually changes between kinetic and potential.

Figure 4.17 Total energy vs time.

Worked example

1 A pendulum bob of mass 200 g is oscillating with amplitude 3 cm and frequency 0.5 Hz. How much KE will the bob have as it passes through the origin?

Solution

1 Since the bob has SHM, $KE_{max} = \frac{1}{2}m\omega^2 x_0^{\ 2}$

where $x_0 = 0.03$ m and $\omega = 2\pi f = 2\pi \times 0.5 = \pi$

$KE_{max} = \frac{1}{2} \times 0.2 \times \pi^2 \times (0.03)^2 = \mathbf{8.9 \times 10^{-4}\ J}$

4.3 Forced oscillations and resonance

Assessment statements

4.3.1 State what is meant by *damping*.
4.3.2 Describe examples of damping.
4.3.3 State what is meant by *natural frequency of vibration* and *forced oscillations*.
4.3.4 Describe graphically the variation with forced frequency of the amplitude of vibration of an object close to its natural frequency of vibration.
4.3.5 State what is meant by *resonance*.
4.3.6 Describe examples of resonance where the effect is useful and where it should be avoided.

Damping

When deriving the equations for KE and PE, we assumed that no energy was lost. In real oscillating systems there is always friction and sometimes also air resistance. The system has to do work against these forces resulting in a loss of energy. This effect is called *damping*.

The suspension of a car. The damper is the red telescopic part.

A car suspension system has many springs between the body and the wheels. Their purpose is to absorb shock caused by bumps in the road.

The car is therefore an oscillating system that would oscillate up and down every time the car went over a bump. As this would be rather unpleasant for the passengers, the oscillations are damped by dampers (wrongly known as shock absorbers).

Light damping

If the opposing forces are small, the result is a gradual loss of total energy. This means that the amplitude of the motion gets slowly less with time. For example, a mass on a spring hanging in the air would have a little damping due to air resistance.

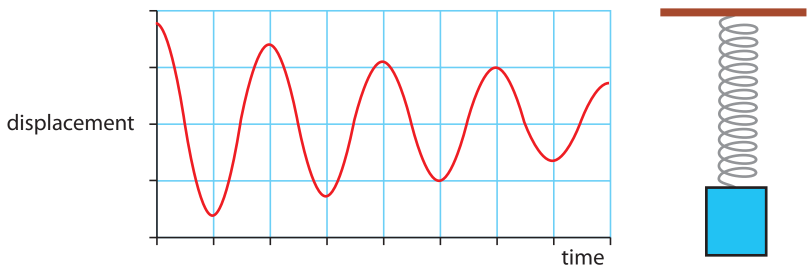


Figure 4.18 Reduction in amplitude due to light damping.

Frequency of damped harmonic motion

You can see from the graph that the frequency does not change as the amplitude gets less. As the motion slows down, the distance travelled gets less, so the time for each cycle remains the same.

If the mass is suspended in water, the damping is greater, resulting in a more rapid loss of energy.

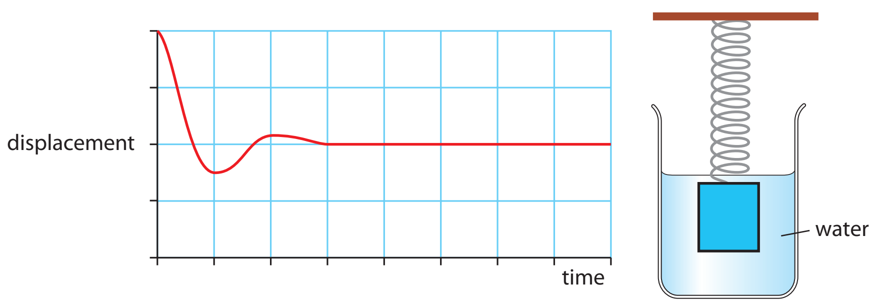


Figure 4.19 Reduction in amplitude due to heavier damping.

Critical damping

Critical damping occurs if the resistive force is so big that the system returns to its equilibrium position without passing through it. This would be the case if the mass were suspended in a thicker liquid such as oil.

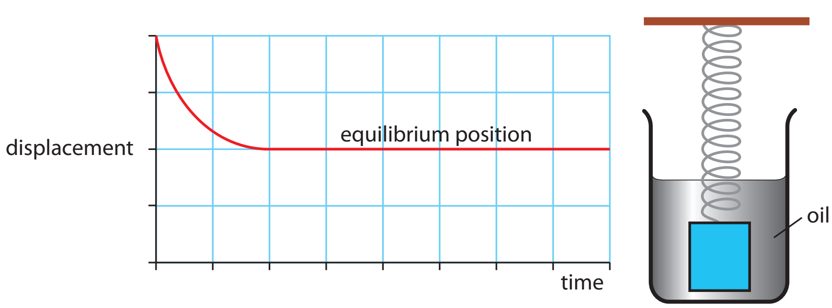


Figure 4.20 Reduction in amplitude due to critical damping.

Resonance

It's possible to shatter a wine glass if you sing at its natural frequency.

In all of the previous examples, a system has been displaced and released, causing an oscillation. The frequency of this oscillation is called the *natural frequency*. If a system is forced to oscillate at a frequency other than the natural frequency, this is called a *forced oscillation.*

Resonance is an increase in amplitude that occurs when an oscillating system is forced to oscillate at its own natural frequency.

For example, when you hit a wine glass with your finger, it vibrates. If you sing at the same frequency, your voice can cause the wine glass to resonate. Sing loud enough and the wine glass will shatter (not many people can do this).

If a spring is pulled down and released, then it will oscillate at its own natural frequency. If the support is oscillated, then the system will be forced to vibrate at another frequency. If the driving frequency is the same as the natural frequency, then resonance occurs.

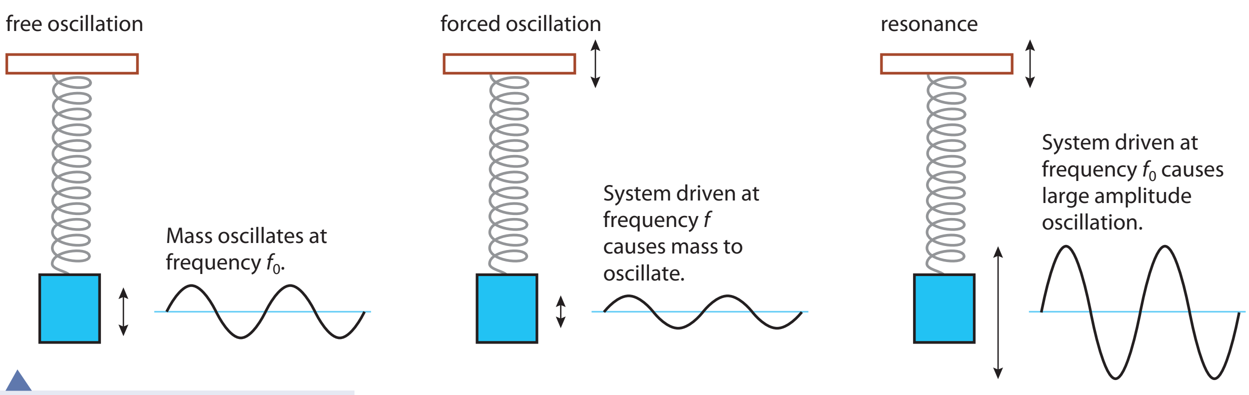


Figure 4.21 The effect of varying driving frequency.

Resonance curve

A graph of the amplitude of oscillation against the driving frequency is called a resonance curve. The sharpness of the peak is affected by the amount of damping in the system.

The radio tuner

When you tune your radio, you are adjusting an electric circuit so that it resonates with the signal of a particular frequency. If the resonance curve for the circuit were not sharp, you would be able to tune into the station over a wide range of frequencies, and would be likely to get interference from other stations.

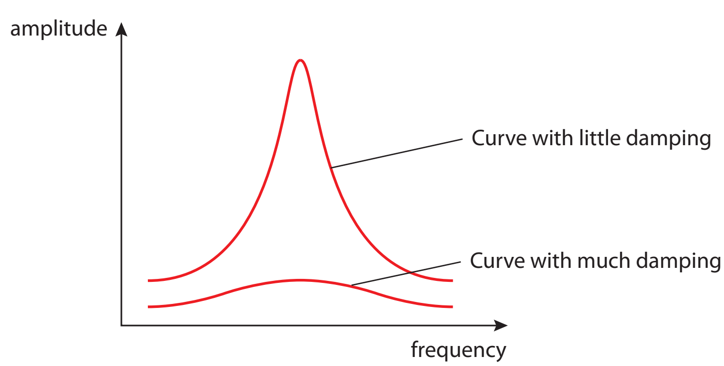


Figure 4.22 Graph of amplitude vs frequency for different levels of damping.

Phase

If we take two identical pendulum bobs, displace each bob to the right and release them at the same time, then each will have the same displacement at the same time. We say the oscillations are *in phase*. If one is pulled to the left and the other to the right, then they are *out of phase*.

Figure 4.23 The pendulum bobs are in phase when they swing together.

This can be represented graphically:

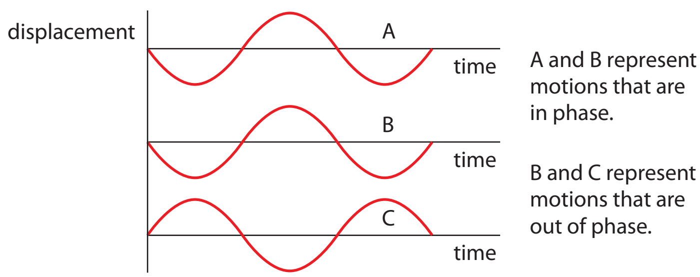


Figure 4.24 Displacement–time graphs for bodies in and out of phase.

Riding a horse
When riding a horse it is important to stay in phase with the horse. If you are out of phase, then you will be coming down when the horse is going up, resulting in an uncomfortable experience. If the horse goes up and down too fast, then it can be very difficult to stay in phase. A mechanical horse is more difficult to ride; you can only accelerate downwards at $9.8\ \mathrm{m\,s^{-2}}$, so if the horse accelerates down too fast then you can't keep up with it.

Phase difference

The phase difference is represented by an angle (usually in radians). We can see from the previous graphs that if two oscillations are completely out of phase then the graphs are displaced by an angle π. We say the *phase difference* is π.

When juggling balls (or oranges) they go up and down at different times – they are out of phase.

Worked example

A ball is sitting on a platform oscillating with amplitude 1 cm at a frequency of 1 Hz. As the frequency is increased, the ball starts to lose contact with the platform. At what frequency does this take place?

Solution

The ball will lose contact when the acceleration of the platform is greater than $9.8\ \mathrm{m\,s^{-2}}$.

Using the formula $a = -\omega^2 x_0$

$$\omega = \sqrt{9.8/0.01} = 31.3\ \mathrm{rad\,s^{-1}}$$

$$f = \frac{\omega}{2\pi} = \mathbf{5\ Hz}$$

Experimental measurement of oscillations

The frequency at which a child oscillates on a swing is low enough to measure using a stopwatch (although to be accurate, you should use some sort of marker, so you can easily judge when the child has made a complete cycle). Higher frequency oscillations are not possible to measure in this way but can be measured using electronic sensors. Here are some examples of how you could make those measurements.

Photo gate

A photo gate sends a signal to a computer each time something passes through it. If a vibrating object passes through the gate each cycle then the computer can calculate the time period of the oscillation.

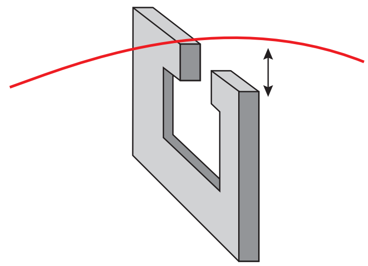


Figure 4.25 A photo gate is used to measure the time period of a vibrating elastic string.

The computer will record each time the string passes through the gate. The time period is the time between the first pass and the third pass. Depending on the software used, it may be possible for the computer to calculate and display the frequency.

Force sensor

When a pendulum swings, the tension in the string varies with time. A force sensor can be used to measure the tension, enabling you to plot a graph of tension *vs* time on the computer. The frequency is calculated from the graph.

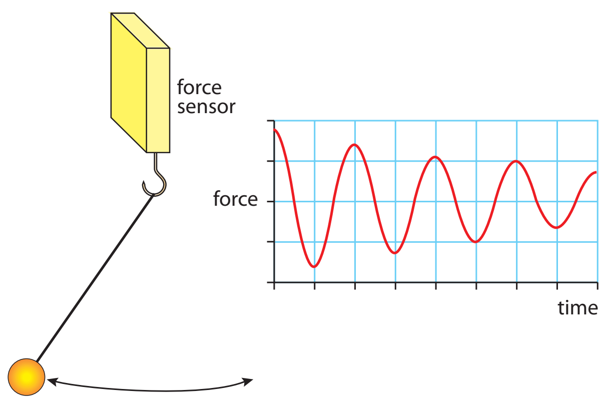


Figure 4.26 The maximum force gets less as the amplitude gets less.

With this method it is also possible to see the damping of the motion.

Position sensor

To measure an oscillation using a position sensor, the oscillating body must move backwards and forwards (or up and down) in front of the sensor. The sensor sends out a sound that is reflected off the object back to the sensor. By measuring the time taken for the sound to reflect back from the object, the computer can calculate the distance between the sensor and the object. This method has the advantage of not disturbing the motion, but the object must be big enough for the sensor to detect it.

4.4 Wave characteristics

Assessment statements

4.4.1 Describe a wave pulse and a continuous progressive (travelling) wave.
4.4.2 State that progressive (travelling) waves transfer energy.
4.4.3 Describe and give examples of transverse and of longitudinal waves.
4.4.4 Describe waves in two dimensions, including the concepts of wavefronts and of rays.
4.4.5 Describe the terms *crest*, *trough*, *compression* and *rarefaction*.
4.4.6 Define the terms *displacement*, *amplitude*, *frequency*, *period*, *wavelength*, *wave speed* and *intensity*.
4.4.7 Draw and explain displacement–time graphs and displacement – position graphs for transverse and for longitudinal waves.
4.4.8 Derive and apply the relationship between wave speed, wavelength and frequency.
4.4.9 State that all electromagnetic waves travel with the same speed in free space, and recall the orders of magnitude of the wavelengths of the principal radiations in the electromagnetic spectrum.

The word *wave* was originally used to describe the way that a water surface behaves when it is disturbed. We use the same model to explain sound, light and many other physical phenomena. This is because they have some similar properties to water waves, so let's first examine the way water waves spread out.

If a stone is thrown into a pool of water, it disturbs the surface. The disturbance spreads out or *propagates* across the surface, and this disturbance is called a *wave*. Observing water waves, we can see that they have certain basic properties (in other words, they do certain things).

Reflection

If a water wave hits a wall, the waves reflect.

Sea waves reflect off a cliff.

Waves change direction as they approach a beach.

Refraction

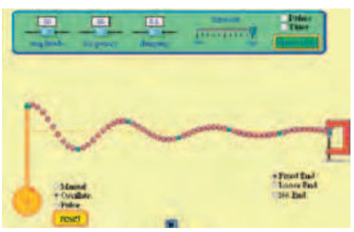
When sea waves approach a beach, they change direction because of the difference in height of different parts of the sea floor. This causes the waves to bend.

Interference

When two waves cross each other, they can add together creating an extra big wave.

Diffraction

When water waves pass through a small opening, the waves spread out.
Anything that reflects, refracts, interferes and diffracts can also be called a wave.



To view the PhET *Waves on a string* simulation, visit heinemann.co.uk/hotlinks, enter the express code 4426P and click on Weblink 4.2.

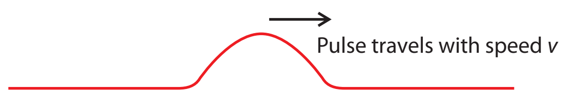
One-dimensional waves

The next step is to derive a model for wave motion and use it to help us understand why waves behave in the way that they do. However, since water waves are two-dimensional, they are not the easiest waves to start with. We will begin by looking at two examples of one-dimensional waves: waves in a string and waves in a spring.

Wave pulse in a string

If a string held between two people is displaced (flicked), a disturbance can be seen to travel from one end to the other. This is called a *wave pulse*.

Figure 4.27 A wave pulse.



We can see that the pulse travels with a certain speed – this is called the *wave speed*.

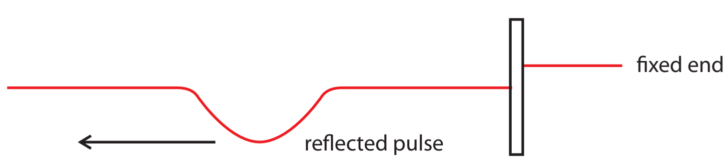
Wave speed is the distance travelled by the wave profile per unit time.

Note: No part of the string actually moves in the direction of the wave velocity – in fact, each particle in the string moves at right angles to the direction of wave velocity.

Reflection of a wave pulse

If the pulse meets a fixed end (e.g. a wall), it exerts an upward force on the wall. The wall being pushed up, pushes back down on the string sending an inverted reflected pulse back along the string.

Figure 4.28 A reflected pulse.



Interference of wave pulses

If two pulses are sent along a string from each end, they will cross each other in the middle of a string.

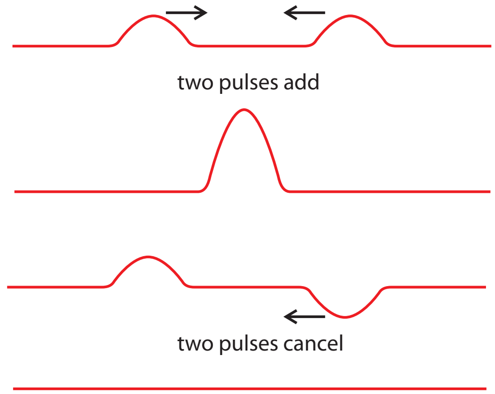


Figure 4.29 The resultant wave is the sum of the individual waves.

Since waves in a string do not spread out, they cannot diffract or refract. You would have to observe the 2D equivalent, waves in a rubber sheet, to see this.

Transfer of energy

It can be seen that as the string is lifted up it is given PE. This PE is transferred along the string. A wave can therefore be thought of as a transfer of energy. There is in fact so much energy transferred by waves in the sea that they can be used to produce electricity.

Continuous waves in a string

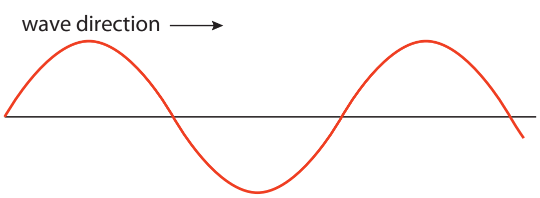


Figure 4.30 The 'sine shape' or profile moves along the string with the wave speed.

If the end of a string is moved up and down with simple harmonic motion of frequency f, a series of pulses moves along the string in the shape of a sine curve, as in Figure 4.30.

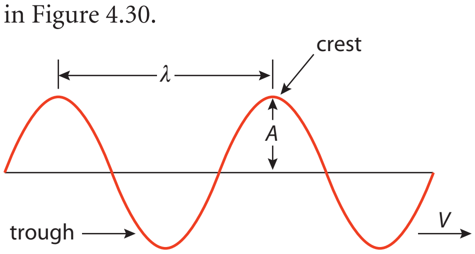


Figure 4.31 The quantities used to define a wave.

Amplitude (A)
The maximum displacement of the string from the equilibrium position.

Wave speed (v)
The distance travelled by the wave profile per unit time.

Wavelength (λ)
The distance between two consecutive crests or any two consecutive points that are in phase.

Frequency (f)
The number of complete cycles that pass a point per unit time.

Period (T)
Time taken for one complete wave to pass a fixed point ($T = 1/f$)

Phase
The phase is a quantity that tells us whether parts of a wave go up and down at the same time or not.

Transverse waves

Figure 4.32 Transverse wave

A wave in a string is an example of a transverse wave. The direction of disturbance is perpendicular to the direction that the wave profile moves.

Stringed instruments

When you pluck the string of a guitar, a wave reflects backwards and forwards along the string. The vibrating string creates the sound that you hear. The pitch of the note is related to the frequency of the string (high pitch = high frequency).

- Why are the low notes thick strings?
  The speed of the wave is inversely related to the mass per unit length of the string. Thick strings have a greater mass per unit length, so the wave will travel more slowly in a thick string. If we rearrange the formula $v = f\lambda$, we find that $f = \frac{v}{\lambda}$ so reducing the wave speed will reduce the frequency of the wave.
- Why does shortening the string make the note higher?
  Shortening the string reduces the wavelength of the wave. According to the formula $f = \frac{v}{\lambda}$, reducing the wavelength will increase the frequency.
- Why does tightening the string make the note higher?
  The wave speed is directly related to the tension in the string. Increasing tension increases the wave speed, which, according to the formula $f = \frac{v}{\lambda}$, will increase the frequency of the wave.

Worked example

1 The A string of a guitar vibrates at 110 Hz. If the wavelength is 153 cm, what is the velocity of the wave in the string?

2 A wave in the ocean has a period of 10 s and a wavelength of 200 m. What is the wave speed?

Relationship between *f* and λ

$v = f\lambda$

If the frequency is *f* then the time for the wave to progress one cycle is 1/*f*. In this time the wave has moved forward a distance equal to one wavelength (λ).

$\text{Velocity} = \frac{\text{distance}}{\text{time}}$

$v = \frac{\lambda}{1/f} = f\lambda$

Solution

1 $v = f\lambda$

$f = 110\text{ Hz and } \lambda = 1.53\text{ m}$

● **Examiner's hint:** Change cm to m.

$v = 110 \times 1.53\text{ m s}^{-1}$

$= \mathbf{168.3\text{ m s}^{-1}}$

2 $T = 10\text{ s}$

$f = 1/T\text{ Hz}$

$= 0.1\text{ Hz}$

$v = f\lambda$

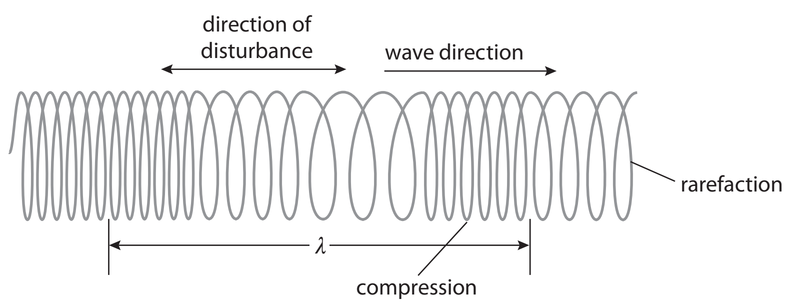
$v = 0.1 \times 200\text{ m s}^{-1}$

$= \mathbf{20\text{ m s}^{-1}}$

Waves in a spring

If a long soft spring (a slinky) is stretched and one end moved back and forth, a compression can be seen to travel along it. Although this may not look like a wave, it *is* transferring energy from one end to the other and so fits the definition.

Figure 4.33 The difference between a compression wave in a spring and the transverse wave in a string is the direction of disturbance.



Longitudinal waves

Figure 4.34 Longitudinal wave.

A compression wave in a slinky is an example of a longitudinal wave. In a longitudinal wave, the disturbance is parallel to the direction of the wave.

Reflection

When the wave in a spring meets a fixed end, it will reflect.

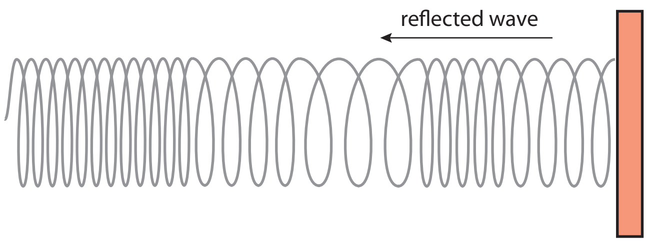


Figure 4.35 A wave in a spring is reflected off a wall.

Earthquake waves
An earthquake is caused when parts of the Earth's crust move against each other. This disturbance causes both longitudinal and transverse waves to spread around the Earth.

Transverse wave
When an earthquake occurs the ground shakes up and down.

Longitudinal wave
The movement in the Earth's crust compresses the rock.

Interference

Although not easy to observe, when two longitudinal waves meet, the displacements superpose in the same way as transverse waves.

Light and sound
Both light and sound are disturbances that spread out, so can be thought of as waves. Light can be polarized (for example, by Polaroid sunglasses) but sound cannot. This is one way to tell that light is transverse and sound is longitudinal.

Distinguishing longitudinal and transverse

A wave is polarized if the displacement is only in one direction.

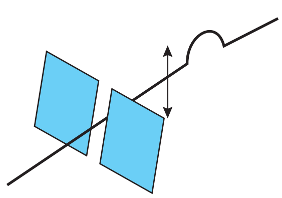


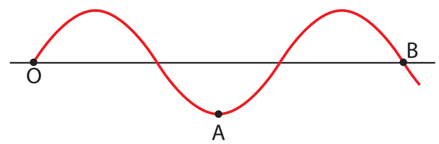
Figure 4.36 A string wave can be polarized by passing through a narrow slit.

The string can only move up and down so a wave in this string will be polarized. To test if the wave is polarized we can place another slit on the string; the wave only passes if the slits are parallel. Only transverse waves can be polarized, so this property can be used to tell if a wave is transverse or longitudinal.

Graphical representation of a wave

There are two ways we can represent a wave graphically, either by drawing a displacement–time graph for one point on the wave, or a displacement–position graph for each point along the wave.

Figure 4.37 A snapshot of a transverse wave.

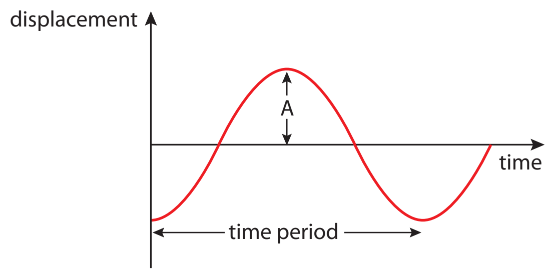


Displacement–time

Consider point A on the transverse wave in Figure 4.37.

Point A is moving up and down with SHM as the wave passes. At present, it is at a minimum of displacement. As the wave progresses past A, this point will move up and then down.

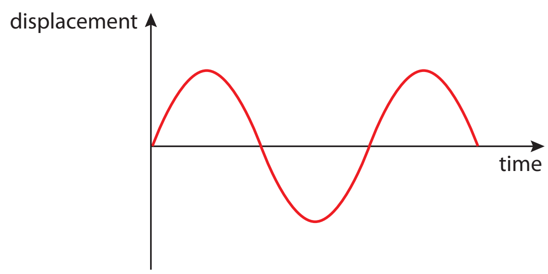
Figure 4.38 The displacement– time graph for point A.



Note: Because the horizontal axis is time, the separation of the peaks represents the time period, not the wavelength.

We can also draw a graph for point B. This point starts with zero displacement then goes up.

Figure 4.39 The displacement–time graph for point B.



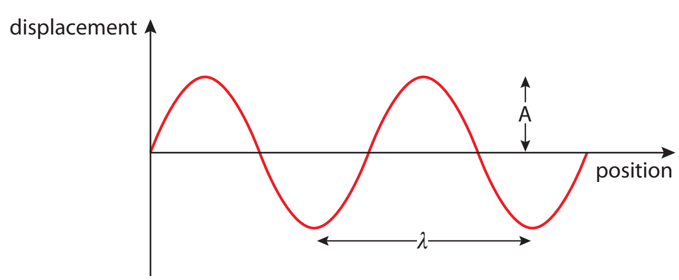
Note: The event that will happen next is to the right on the graph but the part of the wave that will arrive next is to the left on the wave.

Displacement–position

To draw a displacement–position graph, we must measure the displacement of all the points on the wave at one moment in time.

Figure 4.40 shows the graph at the same time as the snapshot in Figure 4.37 was taken. The position is measured from point O.

Figure 4.40 The displacement–position graph for all points at one time.



This is just like a snapshot of the wave – however, depending on the scale of the axis, it might not look quite like the wave.

Longitudinal waves

We can also draw graphs for a longitudinal wave. Consider a chain of balls connected with springs. If the red ball on the left were moved back and forth with SHM, it would send a longitudinal wave along the chain.

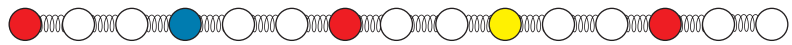


Figure 4.41 A line of balls joined by springs.

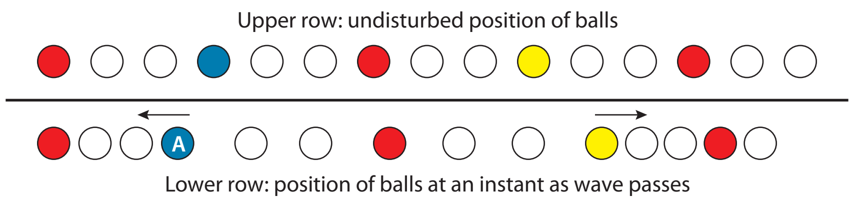


Figure 4.42 A snapshot taken as a wave passes through the chain.

Each ball simply moves back and forth with SHM. Ball A is at present displaced to the left. This ball has negative displacement.

Displacement–time

We can draw a displacement–time graph for ball A starting at the time of the snapshot in Figure 4.42.

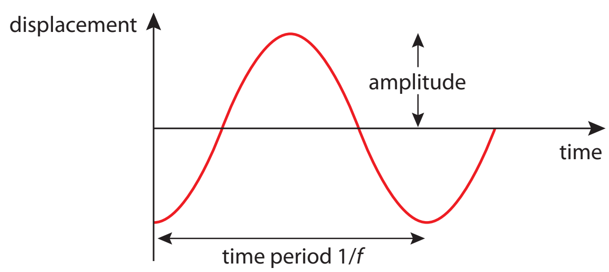


Figure 4.43 The displacement–time graph for point A.

Displacement–position

To draw a displacement–position graph, we must compare the position of each ball with its original position.

The red balls have not moved, so their displacement is 0.

The blue balls have moved to the left, so their displacement is negative.

The yellow ball has moved to the right, so its displacement is positive.

W To see an animated version of this, visit heinemann.co.uk/hotlinks, enter the express code 4426P and click on Weblink 4.3.

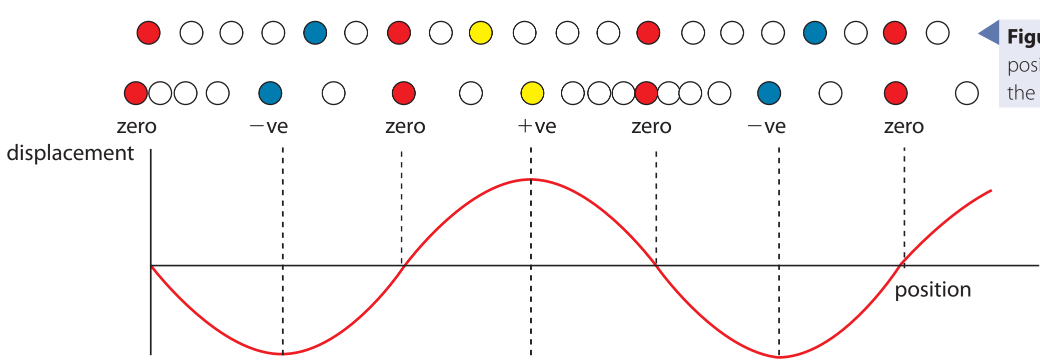
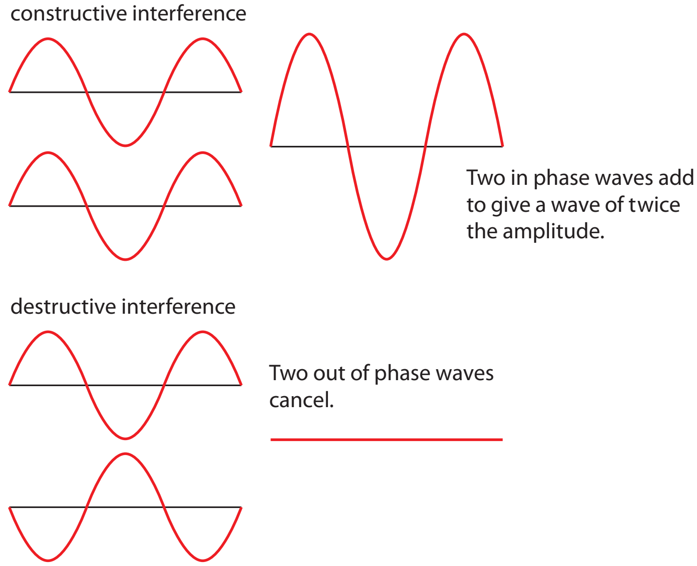


Figure 4.44 The displacement position graph for all points on the wave.

Superposition of one-dimensional waves

When two waves are incident along the same string, we can find the resultant wave by adding the individual displacements.

Figure 4.45 Superposition of waves.



4.5 Wave properties

Assessment statements

4.5.1 Describe the reflection and transmission of waves at a boundary between two media.
4.5.2 State and apply Snell's law.
4.5.3 Explain and discuss qualitatively the diffraction of waves at apertures and obstacles.
4.5.4 Describe examples of diffraction.
4.5.5 State the principle of superposition and explain what is meant by constructive interference and by destructive interference.
4.5.6 State and apply the conditions for constructive and for destructive interference in terms of path difference and phase difference.
4.5.7 Apply the principle of superposition to determine the resultant of two waves

Ripples spreading out in a circle after the surface is disturbed.

Two-dimensional waves

We will now use water waves to model the motion of waves in 2D. If a disturbance is made by a point object in the middle of a tank of water, ripples spread out in circles across the tank. We will use pictures from a computer simulation to show the effect more clearly.

Wavefront

This is a line joining points that are in phase. The straight or circular lines that you can see in the photos are wavefronts.

Rays

Rays are lines drawn to show the direction of the waves – they are always at right angles to the wavefront.

Circular wavefronts

A circular wavefront is produced by a point disturbance. The rays are radial, as they are perpendicular to the wavefronts.

Plane wavefront

Plane wavefronts are produced by an extended disturbance e.g. a long piece of wood dipped into the water, or a point that is so far away that the circles it produces look like straight lines.

A plane wavefront moves towards the beach.

To view the PhET simulation *Wave interference*, visit heinemann.co.uk/hotlinks, enter the express code 4426P and click on Weblink 4.4.

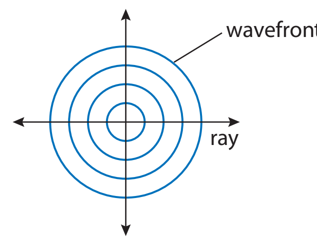


Figure 4.46 A circular wavefront spreading out from a point.

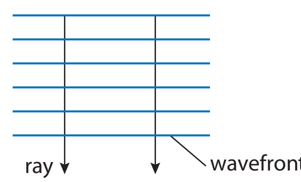


Figure 4.47 Parallel plane wavefronts.

Reflection

When a wave hits a barrier, it is reflected.

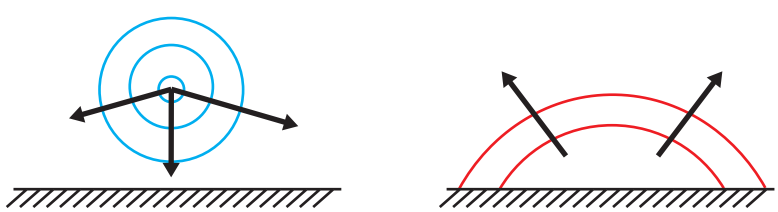
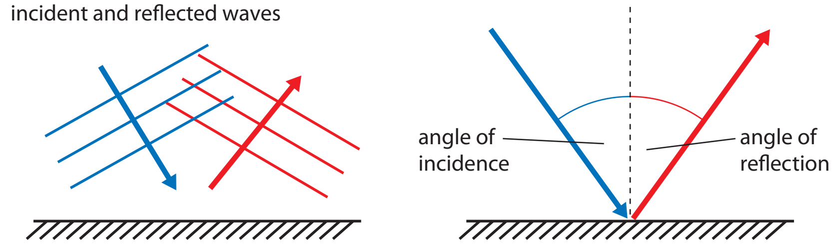
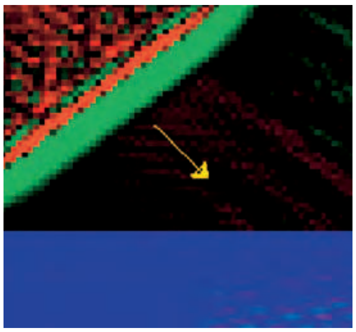


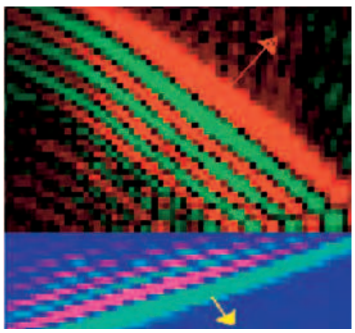
Figure 4.48 Reflection of a circular wavefront.

Notice how the reflected wave appears to originate from somewhere on the other side of the barrier. This is just the same as the appearance of an image of yourself behind a mirror.

Figure 4.49 A plane wavefront is reflected at the same angle that it comes in at.







Rather than measuring the angle that the wavefront makes, it is more convenient to measure the angles that the rays make with a line drawn at 90° to the barrier. This line is called the *normal*.

The laws of reflection

The laws of reflection describe how waves are reflected from barriers.

- The angle of incidence = the angle of reflection.
- The incident and reflected rays are in the same plane as the normal.

Change of medium

Whenever a wave travels from one medium to another, part of the wave is reflected and part transmitted. An example of this is when light hits a glass window: most passes through but a fraction is reflected. So you see a reflection of yourself in the window and someone standing on the other side of the window can see you. The part of the wave that passes through the window is called the *transmitted* part.

To view screenshots showing the reflected and transmitted wave, visit heinemann.co.uk/hotlinks, enter the express code 4426P and click on Weblink 4.5.

Refraction

When a wave passes from one medium to another, its velocity changes. For example, when a water wave passes from deep water into shallow water, it slows down. If the wave hits the boundary between the media at an angle, then the wave also changes direction.

Figure 4.50 Refraction is the change of direction when a wave passes from one medium to another.

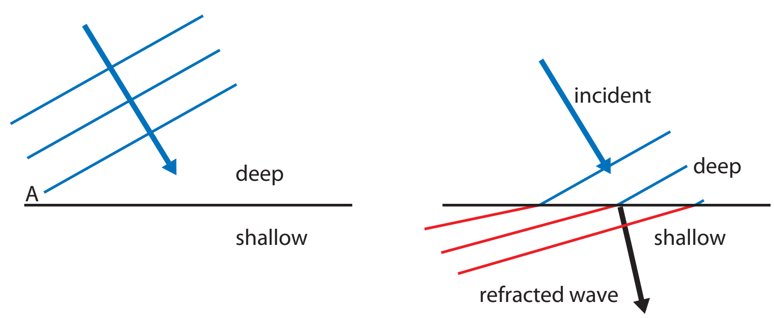
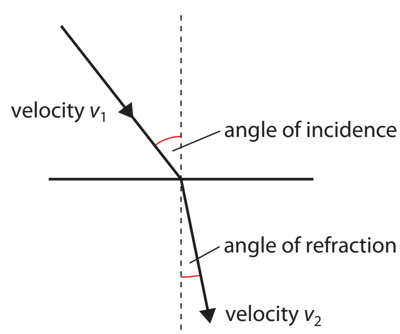


Figure 4.51 Angles of incidence and refraction.



Point A on the incident wave hits the boundary first, so this part of the wave then progresses into the shallow water more slowly. The rest of the wave in the deep water is still travelling fast so catches up with the slow moving part, causing the wavefront to change direction. This is simpler to see if we just draw the rays.

Snell's law

Snell's law relates the angles of incidence and refraction to the ratio of the velocity of the wave in the different media. The ratio of the sine of the angle of incidence to the sine of the angle of refraction is equal to the ratio of the velocities of the wave in the different media.

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2}$$

As can be seen from the example of the bent straw in the photo, light refracts when it passes from one medium to another. The ratio of the velocity of light in the two media is called the *refractive index*.

Light reflected off the straw is refracted as it comes out of the water causing the straw to appear bent.

Worked example

A water wave travelling at 20 m s^{-1} enters a shallow region where its velocity is 15 m s^{-1} (Figure 4.52). If the angle of incidence of the water wave to the shallow region is 50°, what is the angle of refraction?

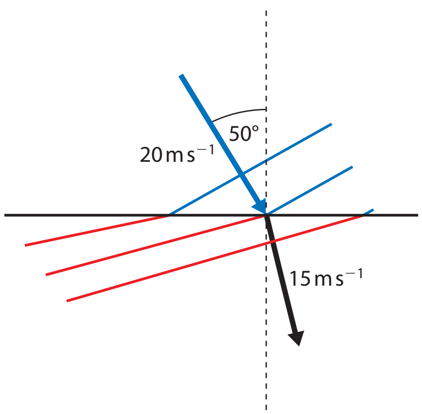


Figure 4.52 Always draw a diagram.

Solution

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2} = \frac{20}{15}$$

so $\sin r = \dfrac{\sin 50°}{1.33} = 0.576$ Applying Snell's law

$\boldsymbol{r = 35.2°}$

Refractive index

When light travelling in air is refracted by an optical medium, the ratio sin *i*/sin *r* is called the refractive index of the medium. If the refractive index is large it means that the light is refracted by a large angle.

Material	Refractive index
Water	1.33
Glass	1.50
Diamond	2.42

Exercises

Use the refractive indices in the table to solve the following problems.

10 Light travelling through the air is incident on the surface of a pool of water at an angle of 40°. Calculate the angle of refraction.

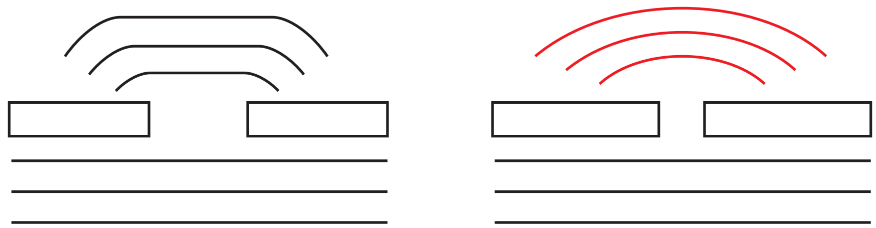
11 Calculate the angle of refraction if a beam of light is incident on the surface of a diamond at an angle of 40°.

12 If the velocity of light in air is 3×10^8 m s^{-1}, calculate its velocity in glass.

Diffraction

Diffraction takes place when a wave passes through a small opening. If the opening is very small, then the wave behaves just like a point source as shown below.

Figure 4.53 If the opening is a bit bigger then the effect is not so great.

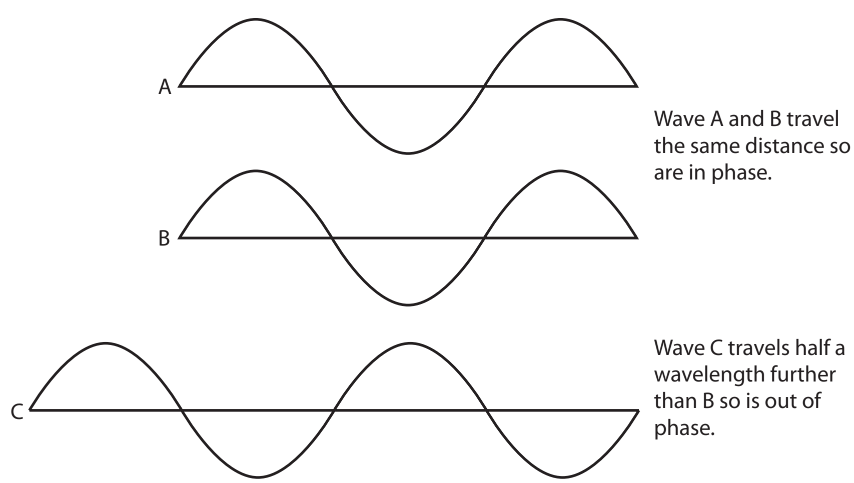


Water waves diffracting through two different sized openings. The waves are diffracted more through the narrower opening.

Interference

When two-dimensional waves interfere, the phase difference between the two waves is different in different places. This means that in some places the waves add and in other places they cancel. This can be seen in the picture below. This shows two sources producing waves of the same frequency.

Figure 4.54 How phase difference is related to path difference.



If the path difference is a whole number of wavelengths, then the waves are *in phase.*

If the path difference is an odd number of half wavelengths then the waves are *out of phase.*

The effect of interference in two dimensions can be seen in Figure 4.55.

Identical waves from A and B spread out across the surface. At X, the waves from A and B have travelled the same distance, so are in phase and add together. At Y, the wave from B has travelled half a wavelength more then the wave from A, so the waves are out of phase and cancel out.

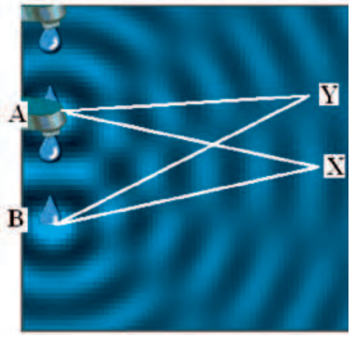


Figure 4.55 Interference effects seen in the PhET simulation. To view this, visit heinemann.co.uk/hotlinks, enter the express code 4426P and click on Weblink 4.6.

Phase angle

If the waves are completely out of phase then phase angle $= \pi$.

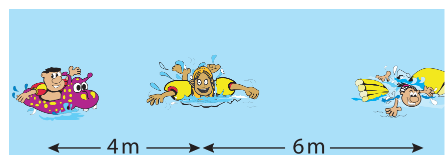
If out of phase but not completely out of phase, then the phase angle can be calculated from the path difference.

If path difference $= d$ then phase angle $\varphi = \dfrac{2\pi d}{\lambda}$

Worked example

Two boys playing in a pool make identical waves that travel towards each other. The boys are 10 m apart and the waves have a wavelength 2 m. Their little sister is swimming from one boy to the other. When she is 4 m from the first boy, will she be in a big wave or a small wave?

Figure 4.56 A diagram always helps, no matter how silly it is.



Solution

The waves from the boys will interfere when they meet, if the girl is 4 m from the first boy, then she must be 6 m from the other. This is a path difference of 2 m, one whole wavelength. The waves are therefore in phase and will add.

Exercise

13 Two wave sources A and B produce waves of wavelength 2 cm. What is the phase angle between the waves at
- **(a)** a point C distance 6 cm from A and 6.2 cm from B?
- **(b)** a point D distance 8 cm from A and 7 cm from B?
- **(c)** a point E distance 10 cm from A and 11.5 cm from B?

Examples of waves

Light

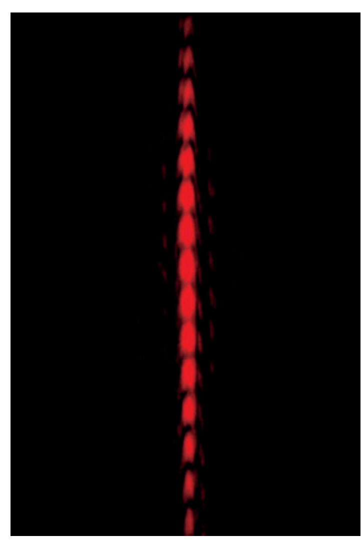
It is worth having a more detailed look at the wave properties of light. We have seen examples of how light reflects and refracts, and if light is a wave, then it must also interfere and diffract.

- Diffraction of light

We have seen that if a wave passes through an opening that is about the same size as its wavelength then it will spread out. If a beam of light passes through a narrow slit (close to the wavelength of light – around 500 nm), it will spread out.

- Interference of light

Waves only interfere if they have the same frequency and similar amplitude. If we take a source of light and split it into two we can create two identical (or *coherent*) wave sources. If the waves from these sources overlap, then areas of bright and dark are created, where the waves interfere constructively and destructively.



The combined effect of diffraction and interference causes this pattern of dots when laser light passes through a pair of narrow slits.

- Polarization

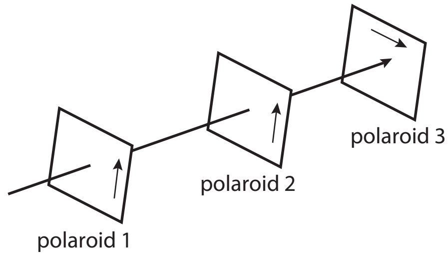
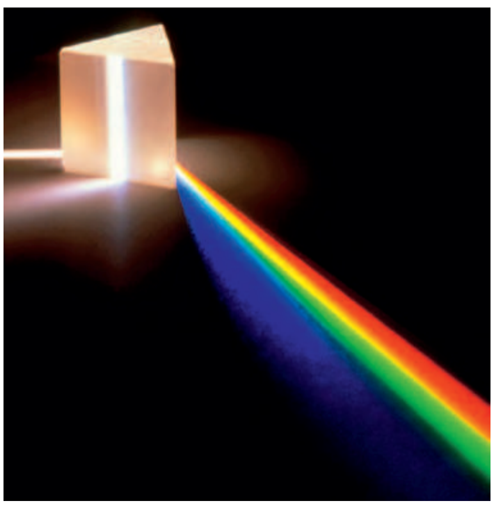


Figure 4.57 The light passes through polaroids 1 and 2, which have the same alignment, but not 3.

When light passes through polaroid sunglasses, it becomes polarized in one direction. We can test to see if the light is polarized by taking a second piece of polaroid and rotating it in front of the sunglasses. As we rotate the polaroid we find that the polarized light can only pass when the second piece is in the same orientation as the first.

- Wavelength and amplitude of light

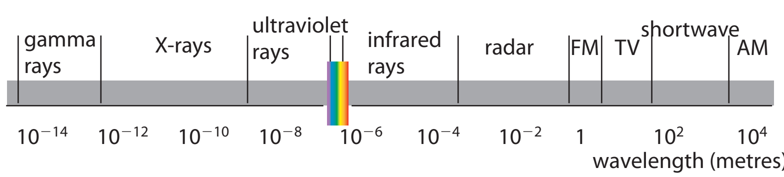
Light is an electromagnetic (EM) wave; that is a propagation of transverse disturbance in an electric and magnetic field. Unlike the other types of waves considered here, EM waves can travel through a vacuum. As with all waves, light waves have wavelength and amplitude. The wavelength of light can vary from 400 nm to 800 nm, with different wavelengths having different colours. White light is made up of all colours mixed together, but if white light is passed through a prism, the different colours are split up, forming a spectrum. This happens because each wavelength refracts by a different amount, and therefore a different angle. This is what happens when a rainbow is formed.



White light can be split up into its component colours by passing it through a prism.

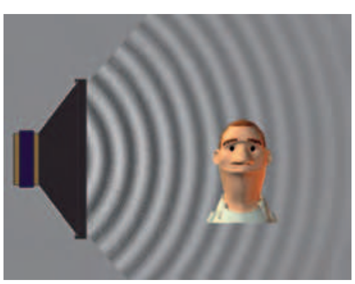
When we say light is a wave we mean it has the same properties as a wave. Does this mean it actually *is* a wave?

Visible light is just one small part of the complete EM spectrum. The full range of wavelength is from 10^{-14}m to 10^{4}m. Each part of the spectrum has different properties and a different name, as illustrated in the diagram below.



The amplitude of light is related to its brightness. The brightness of light is how we perceive light. The physical quantity that measures it is the light *intensity*. This is proportional to the square of the amplitude.

The speed of EM waves in a vacuum is $2.99 \times 10^{8}\ \mathrm{m\,s^{-1}}$.



To view the Phet sound waves simulation, visit heinemann.co.uk/hotlinks, enter the express code 4426P and click on Weblink 4.7.

Sound

- Reflection

If you shout in front of a cliff, the sound reflects back as an echo. In fact any wall is a good reflector of sound, so when speaking in a room, the sound is reflected off all the surfaces This is why your voice sounds different in a room and outside.

- Refraction

When sound passes from warm air into cold air, it refracts. This is why sounds carry well on a still night.

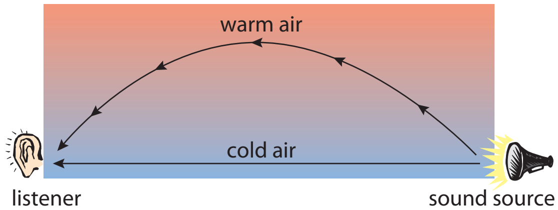


Figure 4.58 Sound refracts through layers of air.

The sound travels to the listener by two paths, one direct and one by refraction through the layers of air. This results in an extra loud sound.

- Diffraction and interference

Because sound reflects so well off all surfaces, it is very difficult to do sound experiments in the laboratory. This makes it difficult to observe sound diffracting and interfering.

A room with no echo is called an anechoic chamber, and these rooms are used for experimenting with sound waves.

Sound spreads out when passing through small openings, around obstacles and through doorways. However, the effects are often owing to multiple reflections rather than diffraction.

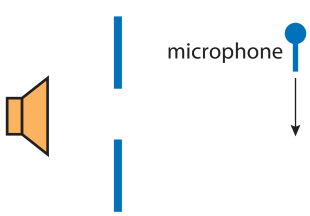


Figure 4.59 The microphone picks up sound owing to diffraction.

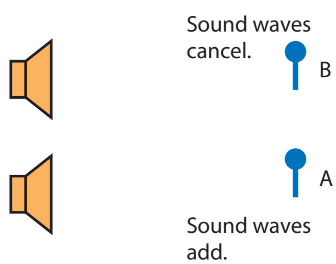
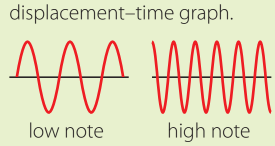


Figure 4.60 Owing to interference, the sound is loud at A but quiet at B.

Sound

Sound is created when the pressure of air is is varied. This change in pressure spreads out as a longitudinal wave. When a sound wave meets a microphone, it causes it to vibrate. The microphone then changes this vibration to an electrical signal that can be used to plot a graph. The graph that we see is a displacement–time graph.



Sound has the properties of a wave, so that means we can use our wave theory to model sound. Sound is a propagation of disturbance in air pressure. Sound is an example of a longitudinal wave. The speed of sound in air is $330\ \text{m s}^{-1}$.

- Frequency and amplitude of sound

Different frequency sound waves have different pitch (that is, a high note has a high frequency). The loudness of a sound is related to the amplitude of the wave.