The law of gravitation

This chapter will introduce you to one of the fundamental laws of physics: Newton's law of gravitation. The law of gravitation makes it possible to calculate the orbits of the planets around the sun, and predicts the motion of comets, satellites and entire galaxies. Newton's law of gravitation was published in his monumental *Philosophiae Naturalis Principia Mathematica*, on 5 July 1686. Newton's law of gravitation has had great success in dealing with planetary motion.

Objectives

By the end of this chapter you should be able to:

- appreciate that there is an attractive force between any two point masses that is directed along the line joining the two masses, $F = G\frac{M_1 M_2}{r^2}$;
- state the definition of *gravitational field strength*, $g = G\frac{M}{r^2}$.

Newton's law of gravitation

We have seen that Newton's second law implies that, whenever a mass moves with acceleration, a force must be acting on it. An object falling freely under gravity is accelerating at 9.8 m s^{-2} and thus experiences a net force in the direction of the acceleration. This force is, as we know, the weight of the mass. Similarly, a planet that revolves around the sun also experiences acceleration and thus a force is acting on it. Newton hypothesized that the force responsible for the falling apple is the same as the force acting on a planet as it revolves around the sun. The conventional weight of a body is nothing more than the gravitational force of attraction between that body and the earth.

▶ Newton proposed that the attractive force of gravitation between two *point* masses is given by the formula

$$F = G\frac{M_1 M_2}{r^2}$$

where M_1 and M_2 are the masses of the attracting bodies, r the separation between them and G a new constant of physics called Newton's constant of universal gravitation. It has the value $G = 6.667 \times 10^{-11}$ N m^2 kg^{-2}. The direction of the force is along the line joining the two masses.

This formula applies to point masses, that is to say masses which are very small (in comparison with their separation). In the case of objects such as the sun, the earth, and so on, the formula still applies since the separation of, say, the sun and a planet is enormous compared with the radii of the sun and the planet. In addition, Newton proved that for bodies which are *spherical* and of uniform density one can assume that the entire mass of the body is concentrated at the centre – as if the body *is* a point mass.

Figure 9.1 shows the gravitational force between two masses. The gravitational force is always attractive.

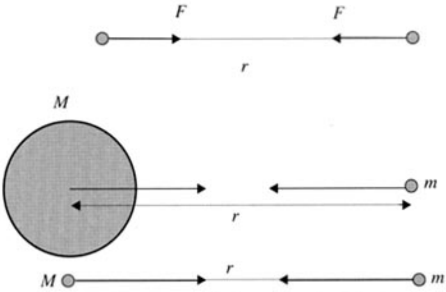

Figure 9.1 The mass of the spherical body to the left can be thought to be concentrated at the centre.

The force on each mass is the same. This follows both from the formula as well as from Newton's third law.

Example questions

Q1

Find the force between the sun and the earth.

Answer

The average distance between the earth and the sun is $R = 1.5 \times 10^{11}$ m. The mass of the earth is 5.98×10^{24} kg and that of the sun 1.99×10^{30} kg. Thus

$F = 3.5 \times 10^{22}$ N

Q2

If the distance between two bodies is doubled, what happens to the gravitational force between them?

Answer

Since the force is inversely proportional to the square of the separation, doubling the separation reduces the force by a factor of 4.

We said that the force we ordinarily call the weight of a mass (i.e. mg) is actually the force of gravitational attraction between the earth of mass M_e and the mass of the body in question. The mass of the earth is assumed to be concentrated at its centre and thus the distance that goes in Newton's formula is the radius of the earth, R_e (see Figure 9.2).

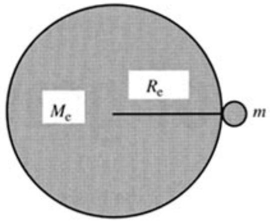

Figure 9.2 The gravitational force due to a spherical uniform mass is the same as that due to an equal point mass concentrated at the centre.

Therefore, we must have that

$$G\frac{M_e m}{R_e^2} = mg$$
$$\Rightarrow g = \frac{GM_e}{R_e^2}$$

This is an extraordinary result. It relates the acceleration of gravity to the mass and radius of the earth. Thus, the acceleration due to gravity on the surface of Jupiter is

$$g = \frac{GM_J}{R_J^2}$$

Example questions

Q3

Find the acceleration due to gravity (the gravitational field strength) on a planet 10 times as massive as the earth and with radius 20 times as large.

Answer

From

$$g = \frac{GM}{R^2}$$

we find

$$\begin{aligned} g &= \frac{G(10M_e)}{(20R_e)^2} \\ &= \frac{10GM_e}{400R_e^2} \\ &= \frac{1}{40}\frac{GM_e}{R_e^2} \\ &= \frac{1}{40}g_e \end{aligned}$$

Thus $g = 0.25$ m s^{-2}.

Q4

Find the acceleration due to gravity at a height of 300 km from the surface of the earth.

Answer

$$g = \frac{GM_e}{(R_e + h)^2}$$

where $R_e = 6.38 \times 10^6$ m is the radius of the earth and h the height from the surface. We can now put the numbers in our calculator to find $g = 8.94 \text{ m s}^{-2}$.

The order-of-magnitude arithmetic without a calculator is as follows:

$$\begin{aligned} g &= \frac{GM_e}{(R_e + h)^2} \\ &= \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(6.68 \times 10^6)^2} \\ &\approx \frac{7 \times 6}{50} \times 10 \\ &\approx \frac{42}{5} \\ &\approx 8 \text{ m s}^{-2} \end{aligned}$$

Gravitational field strength

Physicists (and philosophers) since the time of Newton, including Newton himself, wondered how a mass 'knows' about the presence of another mass nearby that will attract it. By the nineteenth century, physicists had developed the idea of a 'field', which was to provide the answer to the question. A mass M is said to create a *gravitational field* in the space around it. This means that when another mass is placed at some point near M, it 'feels' the gravitational field in the form of a gravitational force. (Similarly, an electric charge will create around it an electric field and another charge will react to this field by having an electric force on it.)

We define **gravitational field strength** as follows.

▶ The gravitational field strength at a certain point is the force per unit mass experienced by a small point mass m at that point. The force experienced by a small point mass m placed at distance r from a mass M is

$$F = G\frac{Mm}{r^2}$$

So the gravitational field strength (F/m) of the mass M is

$$g = G\frac{M}{r^2}$$

The units of gravitational field strength are N kg^{-1}.

If M stands for the mass of the earth, then the gravitational field strength is nothing more than the acceleration due to gravity at distance r from the centre of the earth.

The usefulness of the definition of the gravitational field strength is that it tells us something about the gravitational effects of a given mass without actually having to put a second mass somewhere and find the force on it.

The gravitational field strength is a vector quantity whose direction is given by the direction of the force a point mass would experience if placed at the point of interest. The gravitational field strength around a single point mass M is radial, which means that it is the same for all points equidistant from the mass and is directed towards the mass. The same is true outside a uniform spherical mass. This is illustrated in Figure 9.3.

This is to be contrasted to the assumption of a constant gravitational field strength, which would result in the situation illustrated in Figure 9.4. The assumption of constant acceleration of gravity (as, for example, when we treated projectile motion) corresponds to this case.

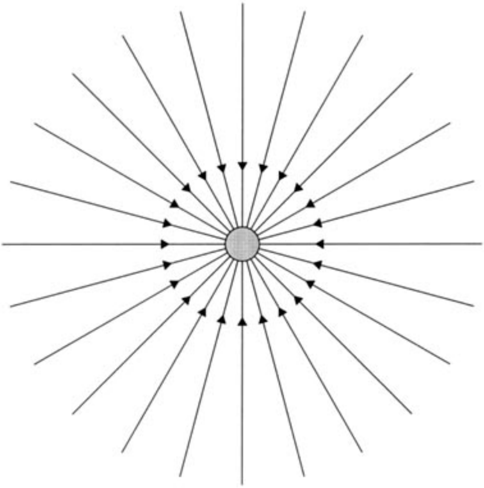

Figure 9.3 The gravitational field around a point (or spherical) mass is radial.

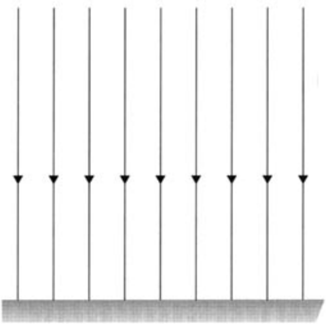

Figure 9.4 The gravitational field above a flat mass is uniform.

Example questions

Q5

Two stars have the same density but star A has double the radius of star B. Determine the ratio of the gravitational field strength at the surface of each star.

Answer

The volume of star A is 8 times that of star B since the radius of A is double. Hence the mass of A is 8 times that of B. Thus

$$\frac{g_A}{g_B} = \frac{GM_A/R_A^2}{GM_B/R_B^2}$$
$$= \frac{M_A}{M_B}\frac{R_B^2}{R_A^2}$$
$$= 8 \times \frac{1}{4}$$
$$= 2$$

Q6

Show that the gravitational field strength at the surface of a planet of density ρ has a magnitude given by $g = \frac{4G\pi\rho R}{3}$.

Answer

We have

$$g = \frac{GM}{R^2}$$

Since

$$M = \rho\frac{4\pi R^3}{3}$$

it follows that

$$g = \frac{G4\pi\rho R^3}{3R^2}$$
$$= \frac{4G\pi\rho R}{3}$$

Questions

1 What is the gravitational force between:
(a) the earth and the moon;
(b) the sun and Jupiter;
(c) a proton and an electron separated by 10^{-10} m?
(Use the data in Appendices 1 and 3.)

2 A mass m is placed at the centre of a thin, hollow, spherical shell of mass M and radius R (see Figure 9.5a).
(a) What gravitational force does the mass m experience?
(b) What gravitational force does m exert on M?
(c) A second mass m is now placed a distance of $2R$ from the centre of the shell (see Figure 9.5b). What gravitational force does the mass inside the shell experience?
(d) What is the gravitational force experienced by the mass outside the shell?

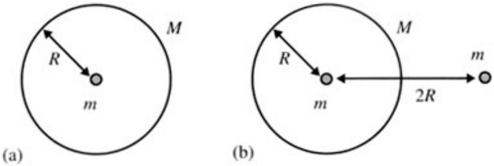

Figure 9.5 For question 2.

3 Stars A and B have the same mass and the radius of star A is 9 times larger than the radius of star B. Calculate the ratio of the gravitational field strength on star A to that on star B.

4 Planet A has a mass that is twice as large as the mass of planet B and a radius that is twice as large as the radius of planet B. Calculate the ratio of the gravitational field strength on planet A to that on planet B.

5 Stars A and B have the same density and star A is 27 times more massive than star B. Calculate the ratio of the gravitational field strength on star A to that on star B.

6 A star explodes and loses half its mass. Its radius becomes half as large. Find the new gravitational field strength on the surface of the star in terms of the original one.

7 The mass of the moon is about 81 times less than that of the earth. At what fraction of the distance from the earth to the moon is the gravitational field strength zero? (Take into account the earth and the moon only.)

8 Point P is halfway between the centres of two equal spherical masses that are separated by a distance of 2×10^9 m (see Figure 9.6). What is the gravitational field strength at:

(a) point P;

(b) point Q?

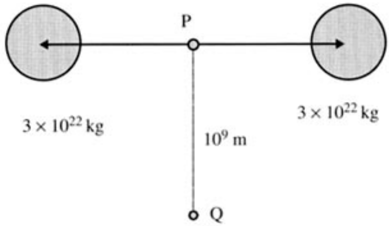

Figure 9.6 For question 8.

HL only

9 Consider two masses. There is a point somewhere on the line joining the masses where the gravitational field strength is zero, as shown in Figure 9.7. Therefore, if a third mass is placed at that point, the net force on the mass will be zero. If the mass is slightly displaced away from the equilibrium position to the left, will the net force on the mass be directed to the left or the right?

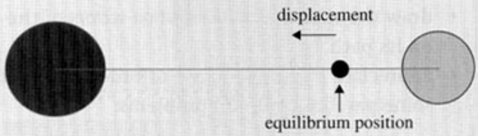

Figure 9.7 For question 9.